K13) $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{1+\cos x}{1+\cos x} \cdot \frac{1-\cos x}{x^2} \overset{AL}{=} \lim_{x\to 0} \underbrace{\frac{1}{1+\cos x}}_{\frac{1}{2}} \cdot \lim_{x\to 0} \frac{1-\cos^2 x}{x^2} =$

$\frac{1}{2} \lim_{x\to 0} \left(\frac{\sin x}{x}\right)^2 \overset{AL}{=} \frac{1}{2}$

K14) $\lim_{x\to 0} \frac{\operatorname{tg} x}{x} \overset{AL}{=} \lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \frac{1}{\cos x} = 1 \cdot 1 = 1$

K15) $\lim_{x\to \frac{\pi}{3}} \frac{\sin(x-\frac{\pi}{3})}{1-2\cos x} \overset{VLSF(I)}{=} \lim_{y\to 0} \frac{\sin y}{1-2\cos(y+\frac{\pi}{3})} = \lim_{y\to 0} \frac{\sin y}{1-2(\cos y \underbrace{\cos\frac{\pi}{3}}_{\frac{1}{2}} - \sin y \underbrace{\sin\frac{\pi}{3}}_{\frac{\sqrt{3}}{2}})}$

$x - \frac{\pi}{3} = y \overset{x\to\frac{\pi}{3}}{\to} 0$, $\neq 0$ na $\mathbb{R}\setminus\{\frac{\pi}{3}\}$

$= \lim_{y\to 0} \frac{\sin y}{1-\cos y + \sqrt{3}\sin y} = \lim_{y\to 0} \frac{\frac{\sin y}{y} \to 1}{\frac{1-\cos y}{y^2} \cdot y + \sqrt{3}\frac{\sin y}{y}}$, $\frac{1-\cos y}{y^2} \to \frac{1}{2}$, $\frac{\sin y}{y} \to 1$

$\overset{AL}{=} \frac{1}{\frac{1}{2}\cdot 0 + \sqrt{3}\cdot 1} = \frac{1}{\sqrt{3}}$.

K16) $\lim_{x\to 0} \frac{\sin 5x - \sin 3x}{\sin x} = \lim_{x\to 0} \frac{\frac{\sin 5x}{5x}\cdot 5x - \frac{\sin 3x}{3x}\cdot 3x}{\frac{\sin x}{x}\cdot x} \overset{AA}{=}$

↑ vykrátíme x a pak bez limitit nedělíme.

$\lim_{x\to 0} \frac{\frac{\sin 5x}{5x}\cdot 5 - \frac{\sin 3x}{3x}\cdot 3}{\frac{\sin x}{x}}$, $\frac{\sin 5x}{5x} \to 1$, $\frac{\sin 3x}{3x} \to 1$, $\frac{\sin x}{x} \to 1$

$\overset{AL}{=}_{(*)} \frac{5-3}{1} = 2$

(*) $\lim_{x\to 0} \frac{\sin ax}{ax} = 1$ $\quad a \neq 0$ ⇐ VLSF(I)

$g(x) = ax = y \overset{x\to 0}{\to} 0$, $x_0 = 0$, $x \in \mathbb{R}\setminus\{0\}$ $\qquad f(y) = \frac{\sin y}{y} \overset{y\to 0}{\to} 1$

K18) $\lim_{x\to 0} \frac{1+\sin x - \cos x}{1-\sin x - \cos x} = \lim_{x\to 0} \frac{\frac{1-\cos x}{x^2}\cdot x^2 + \frac{\sin x}{x}\cdot x}{\frac{1-\cos x}{x^2}\cdot x^2 - \frac{\sin x}{x}\cdot x} =$ ↙ pokrátíme x

$\lim_{x\to 0} \frac{\frac{1-\cos x}{x^2}\cdot x + \frac{\sin x}{x}}{\frac{1-\cos x}{x^2}\cdot x - \frac{\sin x}{x}} \overset{AL}{=} \frac{\frac{1}{2}\cdot 0 + 1}{\frac{1}{2}\cdot 0 - 1} = -1$

K19) $\lim_{x\to 0} \frac{x^2}{\sqrt{1+x\sin x} - \sqrt{\cos x}} = \lim_{x\to 0} \frac{1\cdot(\sqrt{1+x\sin x} + \sqrt{\cos x})}{\frac{1-\cos x}{x^2} + \frac{\sin x}{x}} \overset{(+)AL}{=} \frac{1+1}{\frac{1}{2}+1} = \frac{4}{3}$

(+) $\lim_{x\to 0} \sqrt{1+x\sin x} \overset{VLSF(S)\ spoj.\ \sqrt{}\ v\ 1}{=} \sqrt{1+0} = 1$, $\lim_{x\to 0} \sqrt{\cos x} = \sqrt{1} = 1$

K20) $\lim\limits_{x\to 0} \dfrac{\sqrt{\cos x} - \sqrt[3]{\cos x}}{\sin^2 x} \overset{AL}{=} \lim\limits_{x\to 0} \dfrac{\cos^3 x - \cos^2 x}{\sin^2 x} \cdot \lim\limits_{x\to 0} \dfrac{1}{\underbrace{(\cos x)^{\frac{5}{2}}}_{\to 1} + \dots + \underbrace{(\cos x)^{\frac{5}{3}}}_{\to 1}} \overset{(+)\,AL}{=}$

$\dfrac{1}{6} \cdot \lim\limits_{x\to 0} \cos^2 x \cdot \lim\limits_{x\to 0} \dfrac{\frac{\cos x - 1}{x^2}}{\frac{\sin^2 x}{x^2}} \overset{AL}{=} \dfrac{1}{6} \cdot 1 \cdot \dfrac{-\frac{1}{2}}{1^2} = -\dfrac{1}{12}$

(+) $\lim\limits_{x\to 0} \sqrt{\cos x} = \lim\limits_{x\to 0} \sqrt[3]{\cos x} = 1 \Leftarrow$ spoj. $\sqrt{\ }, \sqrt[3]{\ }$ v 1.

K22) $\lim\limits_{x\to 0} \dfrac{\log(\cos x)}{x^2} = \lim\limits_{x\to 0} \underbrace{\dfrac{\log(\cos x)}{\cos x - 1}}_{\to 1\ (*)} \cdot \dfrac{\cos x - 1}{x^2} \overset{AL}{=} 1 \cdot \left(-\dfrac{1}{2}\right) = -\dfrac{1}{2}$

(*) VLSF(I) $f(y) = \dfrac{\log y}{y - 1} \overset{y\to 1}{\longrightarrow} 1$

$y = g(x) = \cos x \overset{x\to 0}{\longrightarrow} 1$
$\neq 1$ na $(-\pi, \pi) \setminus \{0\}$.

Z,2,4) $\lim\limits_{x\to 0} \dfrac{\sqrt{1+\operatorname{tg} x} - \sqrt{1+\sin x}}{x^3} \overset{AL}{=} \lim\limits_{x\to 0} \dfrac{\operatorname{tg} x - \sin x}{x^3} \cdot \lim\limits_{x\to 0} \dfrac{1}{\underbrace{\sqrt{1+\operatorname{tg} x}}_{\to 1\ (*)} + \underbrace{\sqrt{1+\sin x}}_{\to 1\ (*)}} \overset{AL}{=}$

$\dfrac{1}{2} \lim\limits_{x\to 0} \dfrac{\frac{\sin x}{\cos x} - \sin x}{x^3} \overset{AL}{=} \lim\limits_{x\to 0} \dfrac{1}{\cos x} \cdot \lim\limits_{x\to 0} \left(\dfrac{\sin x}{x} \cdot \dfrac{1 - \cos x}{x^2} \right) \cdot \dfrac{1}{2}$

$= 1 \cdot 1 \cdot \dfrac{1}{2} \cdot \dfrac{1}{2} = \underline{\underline{\dfrac{1}{4}}}$ (*) $\lim\limits_{x\to 0} \sqrt{1+\operatorname{tg} x} = \lim\limits_{x\to 0} \sqrt{1+\sin x} = 1$ $\leftarrow$ VLSF(S) spoj. $\sqrt{\ }$ v 1.

$\boxed{\alpha > -2}$

Pi: $\lim\limits_{x\to 0_+} \dfrac{\operatorname{tg}\left(\log\left(\cos x^{\frac{2}{3}}\right)\right)_{①}}{\sin^2\left(e^{\log(\cos x)\cdot x^{\alpha}} - 1\right)_{②}} = \lim\limits_{x\to 0_+} \underbrace{\dfrac{\operatorname{tg}(\ )_{①}}{(\ )_{①}}}_{\to 1\ [1]} \cdot \underbrace{\dfrac{(\ )_{①}}{\cos x^{\frac{2}{3}} - 1}}_{\to 1\ [2]} \cdot \underbrace{\dfrac{\cos x^{\frac{2}{3}} - 1}{x^{\frac{4}{3}}}}_{\to -\frac{1}{2}\ [3]} \cdot x^{\frac{4}{3}} \cdot$

$\underbrace{\dfrac{\sin^2(\ )_{②}}{\sin^2(\ )_{②}}}_{\to 1\ [4]} \cdot \underbrace{\left(\dfrac{(\ )_{②}}{\sin(\ )_{②}}\right)^2}_{\to 1\ [5]} \cdot \dfrac{1}{x^{2\alpha}} \cdot \underbrace{\left(\dfrac{\cos x - 1}{\log(\cos x)}\right)^2}_{\to 1\ [6]} \cdot \left(\dfrac{x^2}{\cos x - 1}\right)^2 \cdot \dfrac{1}{x^4}$

$\overset{AL}{=} 1 \cdot 1 \cdot \left(-\dfrac{1}{2}\right) \cdot 1 \cdot 1 \cdot 1 \cdot 4 \cdot \lim\limits_{x\to 0_+} x^{-\frac{8}{3} - 2\alpha} = \begin{cases} 0 : & \alpha \in \left(-2, -\frac{4}{3}\right) \\ -2 : & \alpha = -\frac{4}{3} \\ -\infty : & \alpha > -\frac{4}{3} \end{cases}$

[1] VLSF(I) $f(y) = \dfrac{\operatorname{tg} y}{y} \overset{y\to 0}{\longrightarrow} 1$

$g(x) = y = (\ )_{(1)} \overset{x\to 0_+}{\longrightarrow} 0$
$\neq$
$(\ )_{(1)} = 0 \Leftrightarrow \cos x^{\frac{2}{3}} = 1 \Leftrightarrow x^{\frac{2}{3}} = 2k\pi$
$\Rightarrow (\ )_{(1)} \neq 0$ na $(-\pi, \pi) \setminus \{0\}$

[2] VLSF(I) $f(y) = \dfrac{\log(1+y)}{y} \overset{y\to 0}{\longrightarrow} 1$

$y = \cos x^{\frac{2}{3}} - 1 \overset{x\to 0_+}{\longrightarrow} 0$
$\neq 0$ na $(-\pi, \pi) \setminus \{0\}$

[3] VLSF(I) $f(y) = \dfrac{\cos y - 1}{y^2} \overset{y\to 0}{\longrightarrow} -\dfrac{1}{2}$

$y = x^{\frac{2}{3}} \underset{x\to 0_+}{\longrightarrow} 0$
$\neq 0$ na $\mathbb{R} \setminus \{0\}$

[4] VLSF(I) $f(y) = \left(\dfrac{e^y - 1}{y}\right)^{-1} \overset{y\to 0}{\longrightarrow} 1$

$y = \sin^2(\ )_{(2)} \to 0 \Leftarrow$ později (+)
$\neq 0$ $(\ )_{(2)} \notin \{k\pi, k \in \mathbb{Z}\}$ na spoj. okolí 0 $\cos x \neq 1$
kde $(\ )_{(2)} \neq k\pi, k \neq 0$ a $(\ )_{(2)} \neq 0$ pro $x \in (-\pi, \pi) \setminus \{0\}$

[5] VLSF(I) $f(y) = \dfrac{y}{\sin y}$

$y = (\ )_{(2)} \to 0 \Leftarrow (*)$
$\neq 0$ $x \in (-\pi, \pi) \setminus \{0\}$

[6] VLSF(I), $f(y) = \dfrac{y - 1}{\log y} \underset{y\to 1}{\longrightarrow} 1$

$y = \cos x \overset{x\to 0}{\longrightarrow} 1$
$\neq 1$ na $(-\pi, \pi) \setminus \{0\}$

(*) $\lim\limits_{x\to 0} \dfrac{(\ )_{(2)}}{x^{2+\alpha}} \overset{[6]}{=} -2 \Rightarrow$
$(\ )_{(2)} \to 0$.